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THE CHROMATIC NUMBER OF RANDOM GRAPHS

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Let $\chi(G(n,p))$ denote the chromatic number of the random graph $G(n,p)$. We prove that there exists a constant d_0 such that for $np(n) > d_0$, $p(n) \to 0$, the probability that

$$\frac{np}{2\log np}\left(1+\frac{\log\log np-1}{\log np}\right) < \chi(G(n,p)) < \frac{np}{2\log np}\left(1+\frac{30\log\log np}{\log np}\right)$$

tends to 1 as $n \to \infty$.

Let $G(n,p)$ be a random graph with vertex set $[n] = \{1,2,\dots,n\}$ in which each possible edge is present independently with the probability $p = p(n)$. We say that $G(n,p)$ has some property *a.s.* if the probability that it has this property tends to 1 as $n \to \infty$. In this paper we shall consider the asymptotical behaviour of the chromatic number of $G(n,p)$.

The question about the value of $\chi(G(n,p))$ has remained open for over decade until Bollobás in [1] determined $\chi(G(n,p))$ for $p(n)$ which either is a constant or tends to 0 slowly enough i.e. for $p(n) > n^{-1/3+\varepsilon}$ where $\varepsilon > 0$. Matula and Kučera gave in [4] an alternative proof for the case when p is a constant. We shall solve this problem for every function $p(n) \to 0$ such that $np(n)$ is greater than some large constant. Our main result is the following.

Theorem. *There exists a constant d_0 such that if $d = d(n) = np(n) > d_0$ and $p(n) \to 0$ then a.s.*

$$\frac{d}{2\log d}\left(1+\frac{\log\log d-1}{\log d}\right) < \chi(G(n,p)) < \frac{d}{2\log d}\left(1+\frac{30\log\log d}{\log d}\right).$$

The lower bound is an immediate consequence of the fact that if $\varepsilon > 0$, then there exists d_ε such that for $d(n) = np(n) > d_\varepsilon$, the independence number $\alpha(G(n,p))$ of $G(n,p)$ is a.s. less than

$$\alpha(G(n,p)) < \frac{2n}{d}(\log d - \log\log d + 1 - \log 2 + \varepsilon). \tag{1}$$

(This result follows easily from computing the first moment of an appropriate variable.) Thus it is enough to show the second inequality.

Since as we mentioned above the case $p(n) > n^{-1/3+\varepsilon}$ is studied in [1], we shall show our Theorem only for $p(n) < \log^{-7} n$. The method we use works for all $p(n)$ but this technical assumption will simplify calculations significantly.

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Lemma 1. *Let $\varepsilon > 0$ and $k_0 = \lfloor (2n/d) \cdot (\log d - \log\log d + 1 - \log 2 - \varepsilon) \rfloor$. There exists a constant d_ε such that whenever $n \log^{-7} n > np(n) = d(n) > d_\varepsilon$ then, with the probability at least $1 - n^{-2}$, $G(n,p)$ contains a subset with at least $n \log^{-5} d$ vertices which can be properly coloured using $n \log^{-5} d / k_0$ colours.*

Proof. Let $z = \log^{4.9} d$ and $\mathcal{V} = \mathcal{V}(n,p,z)$ denote the partition of $[n]$ onto sets $V_1, \ldots, V_{n/z}$ such that $z - 1 \le |V_i| \le z + 1$ for $i = 1, \ldots, n/z$. We shall call a subset S of $G(n,p)$ *$\mathcal{V}$-disjoint* if no two elements of S are contained in the same set of partition $\mathcal{V}$.

Define $\mathcal{G}_i$ as a graph such that subgraphs induced by $\bigcup_{r=1}^{i} V_r$ in $\mathcal{G}_i$ and $G(n,p)$ are the same and all edges of $\mathcal{G}_i$ which are not contained in $\bigcup_{r=1}^{i} V_r$ are present in this graph independently with the probability p. Let X denote the size of the largest $\mathcal{V}$-disjoint subset contained in $G(n,p)$ which can be properly coloured using $n \log^{-5} d / k_0$ colours, and for $i = 1, 2, \ldots, z$, let X_i be the expectation of the size of the largest $n \log^{-5} d / k_0$ colourable, $\mathcal{V}$-disjoint subset in $\mathcal{G}_i$. Then $X = X_{n/z}$, and setting $X_0 = \mathrm{E}\, X$ we have

$$|X_i - X_{i-1}| \le 1 \qquad \text{for} \quad i = 1, \ldots, n/z\,.$$

Furthermore the sequence $X_0, \ldots, X_{n/z}$ is a martingale known as *Doob's Martingale Process* (see [5]). Hence, from a martingale inequality of Azuma (see [1] or [5]), we get

$$\mathrm{Prob}\,\{|X - \mathrm{E}\, X| \ \ge t\} \le 2 \exp\left(-\frac{zt^2}{2n}\right) ,$$

so

$$\mathrm{Prob}\left\{|X - \mathrm{E}\, X| \ \ge \frac{n}{\log^{6.1} d}\right\} \le 2 \exp\left(-\frac{n}{2 \log^{7.3} d}\right) .$$

Thus, to prove Lemma 1, it is enough to show that the probability that $G(n,p)$ contains a $n \log^{-5} d / k_0$ colourable, $\mathcal{V}$-disjoint subset with more than $n(\log^{-5} d + \log^{-6.1} d)$ elements is greater than $\exp(-n \log^{-7.5} d)$ (this observation was first made by Frieze in [2]).

Let Y be the number of $n \log^{-5} d / k_0$ colourable, $\mathcal{V}$-disjoint subsets of mk_0 elements, where $m = \lceil n \left(\log^{-5} d + \log^{-6.1} d\right) / k_0 \rceil$, which can be split into exactly m independent sets, each of k_0 elements. Then

$$P(Y > 0) \ge \frac{(\mathrm{E}\, Y)^2}{\mathrm{E}\, Y^2}$$

and

$$\frac{\mathrm{E}\,Y^2}{(\mathrm{E}\,Y)^2} \le \prod_{i=1}^{m} \sum_{\substack{k_1,k_2,\ldots,k_{m+1} \\ \sum_{j=1}^{m+1} k_j = k_0}} \frac{\binom{k_0}{k_1}\binom{k_0}{k_2}\cdots\binom{k_0}{k_m}\binom{n/z-(i-1)k_0}{k_{m+1}} z^{k_{m+1}}}{\binom{n/z-(i-1)k_0}{k_0} z^{k_0} (1-p)^{\sum_{j=1}^{m}\binom{k_j}{2}}}$$

$$\le \left[\sum_{l=0}^{k_0} \frac{a_l \binom{n/z-mk_0}{k_0-l} z^{k_0-l}}{\binom{n/z-mk_0}{k_0} z^{k_0}}\right]^m$$

$$\le \left[\sum_{l=0}^{k_0} \frac{a_l}{(n-(k_0+1)mz)^l} \cdot \frac{k_0!}{(k_0-l)!}\right]^m ,$$

where

$$a_l = \sum_{\substack{k_1,k_2,\ldots,k_m \\ \sum_{j=1}^{m} k_j = l}} \binom{k_0}{k_1}\binom{k_0}{k_2}\cdots\binom{k_0}{k_m}(1-p)^{-\sum_{j=1}^{m}\binom{k_j}{2}} .$$

Let k_{i_1}, k_{i_2}, $\ldots$, k_{i_r}, be those from k_1, k_2, $\ldots$, k_m which are greater than $2n\log\log d/d$. Since $\sum_{j=1}^{m} k_j = l \le k_0 < 2n\log d/d$, so $r < \log d$. Thus the number of terms with different sequences $k_{i_1}, k_{i_2}, \ldots, k_{i_r}$ is less than

$$(mk_0)^{\log d} < n^{\log d} . \tag{2}$$

Moreover, for every k', k'' such that $k' \ge k'' \ge 2n\log\log d/d$ and $k' + k'' \le l \le k_0$, we have

$$\frac{\binom{k_0}{k'}\binom{k_0}{k''}(1-p)^{-\binom{k'}{2}-\binom{k''}{2}}}{\binom{k_0}{k'+k''}(1-p)^{-\binom{k'+k''}{2}}} < 1 .$$

Indeed, when $2n\log\log d/d \le k' + k'' \le 0.7k_0$ then

$$\frac{\binom{k_0}{k'}\binom{k_0}{k''}(1-p)^{-\binom{k'}{2}-\binom{k''}{2}}}{\binom{k_0}{k'+k''}(1-p)^{-\binom{k'+k''}{2}}} = \frac{(k_0)_{k''}}{(k_0-k')_{k''}}\binom{k'+k''}{k''}\exp(-pk''k')$$

$$\le \left(\frac{k_0-k''}{k_0-k'-k''}\frac{e(k'+k'')}{k''}\exp(-pk')\right)^{k''} \le \left(\frac{20}{\log d}\right)^{k''} < 1 ,$$

whereas for $k' + k'' \ge 0.7k_0$, $k' \ge k'' \ge 2n\log\log d/d$ we have

$$\frac{\binom{k_0}{k'}\binom{k_0}{k''}(1-p)^{-\binom{k'}{2}-\binom{k''}{2}}}{\binom{k_0}{k'+k''}(1-p)^{-\binom{k'+k''}{2}}} \le 2^{k_0}2^{k_0}\exp(-pk'k'') < 1 .$$

Hence

$$(3)\qquad \binom{k_0}{k_{i_1}}\binom{k_0}{k_{i_2}}\cdots\binom{k_0}{k_{i_r}}(1-p)^{-\sum_{j=1}^{r}\binom{k_{i_j}}{2}} \le \binom{k_0}{l}\exp\left(\frac{l^2d}{2n}\right).$$

Furthermore, for every choice of $\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_s$, one can easily get the following inequality

$$(4)\qquad \sum_{\substack{\bar{k}_1,\bar{k}_2,\ldots,\bar{k}_s\\ \sum_{j=1}^{s}\bar{k}_j=l\\ \max_{1\le j\le s}\bar{k}_j=f}} \binom{k_0}{\bar{k}_1}\binom{k_0}{\bar{k}_2}\cdots\binom{k_0}{\bar{k}_s}(1-p)^{-\sum_{j=1}^{s}\binom{\bar{k}_j}{2}} \le \binom{sk_0}{l}\exp\left(\frac{fld}{2n}\right).$$

Thus, from (2), (3) and (4) we obtain

$$a_l \le \binom{mk_0}{l}\log^{2l}d + n^{\log d}\sum_{i=2n\log\log d/d}^{l}\binom{k_0}{i}\exp\left(\frac{i^2d}{2n}\right)\binom{mk_0}{l-i}\log^{2(l-i)}d.$$

For $2n\log\log d/d \le i \le l$, set

$$b_{i,l} = \binom{k_0}{i}\exp\left(\frac{i^2d}{2n}\right)\binom{mk_0}{l-i}\log^{2(l-i)}d.$$

Then

$$\begin{aligned}\frac{b_{i+1,l}}{b_{i,l}} &= \frac{k_0-i}{i+1}\exp\left(\frac{(2i+1)d}{2n}\right)\frac{l-i}{mk_0-l+i+1}\frac{1}{\log^2 d}\\ (5)\qquad &= \left(\frac{l+1}{i+1}-1\right)\frac{k_0-i}{mk_0-l+i+1}\exp\left(\frac{(2i+1)d}{2n}\right)\frac{1}{\log^2 d}.\end{aligned}$$

Since the second factor of (5) is less than $2\log^6 d/d$ so there exists i_1, such that $b_{i+1,l}/b_{i,l} < 1$ for $i_0 = 2n\log\log d/d < i \le i_1$. Then, if l is large enough, the exponential factor becomes large, and for some i_2 $b_{i+1,l}/b_{i,l} \ge 1$ whenever $i_1 < i \le i_2$. Finally the first factor and the numerator of the second one make $b_{i+1,l}/b_{i,l}$ less than 1 for $i_2 < i \le l$. Hence, $b_{i,l}$ is maximised either when $i = i_0$ or, provided $i_1 < l$, for $i = i_2$.

An upper bound for $b_{i_0,l}$ is given by

$$\begin{aligned}b_{i_0,l} &= \binom{k_0}{i_0}\exp\left(\frac{i_0^2d}{2n}\right)\binom{mk_0}{l-i_0}\log^{2(l-i_0)}d\\ &\le \binom{(m+1)k_0}{l}\log^{2l}d \le \left(\frac{3mk_0\log^2 d}{l}\right)^l.\end{aligned}$$

Let $l = (\alpha + o(1))k_0$ for some constant $0 < \alpha \le 0.9$. Thus, for $i' = 0.99l$, the first factor in (5) tends to a constant, the second is equal $O(1)\log^6 d/d$, whereas from the exponential one we get $d^{1.98\alpha + o(1)}$. Hence $b_{i+1,l}/b_{i,l} < 1$ for $\alpha < 1/1.98$ and $i \le i'$, whereas, when $\alpha > 1/1.98$, we have $b_{i'+1,l}/b_{i',l} \ge 1$. Setting $i' = 0.995l$ one can check in the same way that also for $\alpha = 1/1.98$ $b_{i,l}$ is maximised for either $i = i_0$ or $i_2 > 0.99l$. Moreover for $l \le 0.9k_0$ we have

$$\begin{aligned}
\max\{b_{i,l} : 0.99l \le i \le l\} &\le \binom{2k_0}{l} \exp\left(\frac{l^2 d}{2n}\right) \left(\frac{100emk_0}{l}\right)^{0.01l} \log^{2l} d \\
&\le \left(\frac{2ek_0}{l} \exp\left(\frac{ld}{2n}\right) \left(\frac{n}{l}\right)^{0.01} \log^2 d\right)^l \\
&\le \left(\frac{6k_0 d^{0.92}}{l}\right)^l .
\end{aligned}$$

If $l \ge 0.9k_0$ then $i_2 > i'' = l - ld^{-0.3}$. Indeed, the first factor of $b_{i''+1,l}/b_{i'',l}$ is bounded above by $d^{-0.3}$, the second one is larger than $d^{-1.3}$ and the exponential factor gives us at least $d^{1.8}$. Thus $b_{i''+1,l}/b_{i'',l} > 1$ and so $i_2 > i''$. Furthermore the following inequality holds

$$\begin{aligned}
\max\{b_{i,l} : l - ld^{-0.3} \le i \le l\} &\le \binom{k_0}{l - ld^{-0.3}} \exp\left(\frac{l^2 d}{2n}\right) \left(\frac{emk_0}{ld^{-0.3}}\right)^{ld^{-0.3}} \log^{2ld^{-0.3}} d \\
&\le \binom{k_0}{k_0 - l} \exp\left(\frac{l^2 d}{2n}\right) d^{3k_0 d^{-0.3}} .
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{l=0}^{k_0} \frac{a_l}{(n-(k_0+1)mz)^l} \cdot \frac{k_0!}{(k_0-l)!} \le \sum_{l=0}^{k_0} \left(\frac{2k_0}{n}\right)^l \left(\frac{3mk_0 \log^2 d}{l}\right)^l \\
&+ n^{2\log d} \sum_{l=0}^{0.9k_0} \left(\frac{2k_0}{n}\right)^l \left(\frac{6k_0 d^{0.92}}{l}\right)^l \\
&+ n^{2\log d} d^{3k_0 d^{-0.3}} \sum_{j=0}^{0.1k_0} \frac{k_0!}{(n-(k_0+1)mz)^{k_0}} \frac{n^j}{j!} \binom{k_0}{j} \exp\left(\frac{k_0^2 d}{2n}\right) \exp\left(\frac{(-2jk_0+j^2)d}{2n}\right) \\
&\le \sum_{l=0}^{k_0} \left(\frac{6mk_0^2 \log^2 d}{ln}\right)^l + n^{2\log d} \sum_{l=0}^{0.9k_0} \left(\frac{n}{ld^{1.05}}\right)^l \\
&+ n^{3\log d} d^{3k_0 d^{-0.3}} \left(\frac{k_0}{en(1-\log^{-0.05} d)} \exp\left(\frac{k_0 d}{2n}\right)\right)^{k_0} \sum_{j=0}^{0.1k_0} \left(\frac{k_0 n}{j^2 d^{1.8}}\right)^j .
\end{aligned}$$

All terms of the first sum are less than $\exp\left(6mk_0^2\log^2 d/n\right)$, for d is large enough, the second sum is bounded above by $\exp(nd^{-1.04})$, whereas for the last term we have

$$\frac{k_0}{en(1-\log^{-0.05} d)}\exp\left(\frac{k_0 d}{2n}\right) < \frac{2(\log d - \log\log d + 1)}{en(1-\log^{-0.05} d)}\,\frac{ede^{-\varepsilon}}{2\log d} < 1$$

and

$$n^{3\log d} d^{3k_0 d^{-0.3}} \sum_{j=0}^{0.1k_0}\left(\frac{k_0 n}{j^2 d^{1.8}}\right)^j \le \exp(3k_0 d^{-0.25} + nd^{-1.3}) \le \exp(nd^{-1.2})\,.$$

Thus

$$\sum_{l=0}^{k_0}\frac{a_l}{(n-(k_0+1)mz)^l}\frac{k_0!}{(k_0-l)!}$$
$$\le n^{3\log d}\exp\left(\frac{6mk_0^2\log^2 d}{n}\right) + n^{2\log d}\exp\left(nd^{-1.04}\right) + \exp\left(nd^{-1.2}\right)$$
$$\le \exp\left(\frac{7mk_0^2\log^2 d}{n}\right)\,.$$

and, finally, we arrive at

$$\mathrm{Prob}(Y>0) \ge \frac{\mathrm{E}\,Y^2}{(\mathrm{E}\,Y)^2} \ge \left[\exp\left(\frac{7mk_0^2\log^2 d}{n}\right)\right]^{-m}$$
$$\ge \exp\left(-7n\log^2 d\left(\frac{k_0 m}{n}\right)^2\right) > \exp(-n\log^{-7.5} d)\,.$$

This completes the proof of Lemma 1. ∎

Lemma 2. *There is a constant d_0 such that for $d = d(n) = np(n)$, where $n\log^{-7} n > d > d_0$, with the probability greater than $1 - o(1) - \log^{-1} d$, more than $n - 2n\log^{-3} d$ vertices of $G(n,p)$ can be properly coloured with less than $\frac{d}{2\log d}\left(1+\frac{29\log\log d}{\log d}\right)$ colours.*

Proof. In the proof we shall use an "expose-and-merge" technique introduced by David Matula (for details see [3] and [4]).

For $A \subset [n]$ define $[A]^2 = \{\{v,w\} : v,w \in A\}$ and let

$$\bar{k}_0 = \frac{2n}{d}(\log d - 29\log\log d + 0.1), \qquad l_0 = n/(\bar{k}_0\log^{33} d).$$

Consider the following algorithm:

Algorithm

$E := \emptyset$;
$F_0 := \emptyset$;
$W_0 := \emptyset$;
for $i = 1$ **to** $\log^{33} d - \log^{30} d$ **do**
begin
choose randomly $A_i \subset [n] \setminus W_{i-1}$ with $|A_i| = n \log^{-28} d$;
define $\mathcal{G}_i$ as the graph with the set of vertices A_i and the set of edges E_i, where the probability that $\{v, w\} \in E_i$ is independent and equal p for each $\{v, w\} \in [A_i]^2$;
choose a family $\{\tilde{\mathcal{R}}^i_1, \tilde{\mathcal{R}}^i_2, \dots, \tilde{\mathcal{R}}^i_{l_0}\}$ of disjoint independent sets from A_i, such that $\sum_{l=0}^{l_0} |\tilde{\mathcal{R}}^i_l| = n \log^{-33} d$ - if it is not possible FAIL ;
$E'_i := E_i \setminus (E_i \cap F_{i-1})$;
$E := E \cup E'_i$;
$F_i := F_{i-1} \cup [A_i]^2$;
$W := W_{i-1} \cup \bigcup_{l=1}^{l_0} \tilde{\mathcal{R}}^i_l$;
end
$\overline{F} := [n]^2 \setminus \bigcup_{i=1}^{\log^{33} d - \log^{30} d} F_i$;
choose $\overline{E} \subset \overline{F}$ in such a way that each $e \in \overline{F}$ belong to $\overline{E}$ with the probability p independently of each other ;
$E := E \cup \overline{E}$;
output E ; $\{\tilde{\mathcal{R}}^1_1, \tilde{\mathcal{R}}^1_2, \dots, \tilde{\mathcal{R}}^{\log^{33} d - \log^{30} d}_{l_0}\}$;
end

Let us observe first that the probability that $\{v, w\} \in E$ is equal p independently for each $\{v, w\} \in [n]^2$, so the graph $\tilde{\mathcal{G}}$ with the set of vertices $[n]$ and the set of edges E may be treated as $G(n, p)$.

Obviously, we may consider each $\mathcal{G}_i$ as $G(\overline{n}, p)$, where $\overline{n} = n \log^{-28} d$. The average degree of such a random graph is given by $\overline{d} = p\overline{n} \log^{-28} d = d \log^{-28} d$ and

$$\overline{k}_0 = \frac{2n}{d}(\log d - 29 \log\log d + 0.1) < \frac{2\overline{n}}{\overline{d}}(\log \overline{d} - \log\log \overline{d} + 0.2) .$$

Thus, from Lemma 1, the probability that $\mathcal{G}_i$ contains no subsets with $n \log^{-33} d < \overline{n} \log^{-5} \overline{d}$ elements which can be properly coloured using $\overline{n} \log^{-5} \overline{d}/\overline{k}_0$ colours is less than n^{-2}, so the probability of FAIL in the Algorithm is less than n^{-1}.

Thus, with the probability at least $1 - o(1)$, the Algorithm finds

$$(\log^{33} d - \log^{30} d) l_0 = \frac{n - n \log^{-3} d}{\overline{k}_0} < \frac{d}{2 \log d}\left(1 + \frac{29 \log\log d}{\log d}\right)$$

disjoint sets $\widetilde{\mathcal{R}}\,^1_1, \widetilde{\mathcal{R}}\,^1_2, \ldots, \widetilde{\mathcal{R}}\,^{\log^{33} d - \log^{30} d}_{l_0}$ such that

$$\sum_i \sum_l |\widetilde{\mathcal{R}}\,^i_l| = (\log^{33} d - \log^{30} d)\, n \log^{-33} d = n - n \log^{-3} d\,.$$

Note however, that although $\widetilde{\mathcal{R}}\,^i_l$ is an independent subset of $\mathcal{G}_i$ it is *not* necessarily independent as a subset of $\widetilde{\mathcal{G}}$. Let X denote the number of edges of $\widetilde{\mathcal{G}}$ contained in $\widetilde{\mathcal{R}}\,^i_l$ for some $1 \le i \le \log^{33} d - \log^{30} d$, $1 \le l \le l_0$. We shall estimate from above the size of X.

Let $\{v, w\} \in [n]^2$ be such that $\{v, w\} \in E$ and $\{v, w\} \in \widetilde{\mathcal{R}}\,^i_l$ for some i, l. Denote by $i(v, w)$ the smallest number i for which $\{v, w\} \subset A_i$ and let $j(v, w)$, $l\,(v, w)$ be such that $\{v, w\} \subset \widetilde{\mathcal{R}}\,^{j(v,w)}_{l\,(v,w)}$. Notice that $i(v, w) < j(v, w)$ since $\{v, w\} \in E$ implies that $\{v, w\}$ is an edge of $\mathcal{G}_{i(v,w)}$ whereas the set $\widetilde{\mathcal{R}}\,^{j(v,w)}_{l\,(v,w)}$ contains no edges of $\mathcal{G}_{j(v,w)}$. Since for all i we have $|W_i| \le n - n \log^{-3} d$, so the probability that for chosen $i(v, w)$, $j(v, w)$, a pair $\{v, w\}$ is contained in both $A_{i(v,w)}$ and $A_{j(v,w)}$ is less than $\left(2 \log^{-25} d\right)^4$. Now observe that for any l, where $1 \le l \le l_0$, each subset of $A_{j(v,w)}$ of $|\widetilde{\mathcal{R}}\,^{j(v,w)}_l|$ elements is equally likely to be chosen as $\widetilde{\mathcal{R}}\,^{j(v,w)}_l$ (i.e. this event depends only on the structure of $\mathcal{G}_{j(v,w)}$ which is symmetric with respect to the labelling of vertices). Moreover, due to (1), we may assume that for all i, l, we have $|\widetilde{\mathcal{R}}\,^i_l| < 2n \log d/d$. Thus, since $|A_{j(v,w)}| = n \log^{-28} d$, the probability that both v, w are in the same set $\widetilde{\mathcal{R}}\,^{j(v,w)}_{l\,(v,w)}$ for some $l\,(v, w)$ is less than $2 \log^{29} d/d$.

So finally we arrive at the following upper bound for the expectation of X

$$\mathrm{E}X \le \binom{n}{2} \binom{\log^{33} d - \log^{30} d}{2} \cdot 16 \log^{-100} d \cdot p \cdot 2 \log^{29} d/d < 8n \log^{-5} d\,.$$

Thus, from Markov's inequality,

$$\mathrm{Prob}(X > 0.5 n \log^{-3} d) < \log^{-1} d\,.$$

Now, for all i, l, delete from $\widetilde{\mathcal{R}}\,^i_l$ all these vertices which belongs to edges of $\widetilde{\mathcal{G}}$ which are contained in $\widetilde{\mathcal{R}}\,^i_l$ and denote the set obtained in this way by $\mathcal{R}^i_l$. Then

$$\sum_{i,l} |\mathcal{R}^i_l| \ge \sum_{i,l} |\widetilde{\mathcal{R}}\,^i_l| - 2X = n - n \log^{-3} d - 2X$$

so

$$\mathrm{Prob}\left(\sum_{i,l} |\mathcal{R}^i_l| \ge n - 2n \log^{-3} d \right) > 1 - o(1) - \log^{-1} d$$

and the assertion follows. ∎

Proof of Theorem. Lemma 2 implies that with the probability at least $1-o(1)-\log^{-1} d$ we can colour $n - 2n\log^{-3} d$ vertices of $G(n,p)$ using only $\frac{d}{2\log d}\left(1+\frac{29\log\log d}{\log d}\right)$ colours. One can easily check that for d large enough a.s. every subgraph of $G(n,p)$ on s vertices, where $s \le 2n\log^{-3} d$, has less than $sd\log^{-2.5} d$ edges. Hence, for large d, we have

$$\text{Prob}\left(\chi(G(n,p)) \le \frac{d}{2\log d}\left(1+\frac{29.5\log\log d}{\log d}\right)\right) > 1 - o(1) - \log^{-1} d\,, \tag{6}$$

and for $d(n) \to \infty$, the assertion follows.

Moreover, Shamir and Spencer proved in [5] that for every $d(n) < \log n$ there exists a function $u_d(n)$ such that a.s.

$$u_d(n) \le \chi(G(n,p)) \le u_d(n) + 4\,.$$

Thus, since from (6) for $d(n)$ greater than some constant d_0 we have

$$\liminf_{n\to\infty} \text{Prob}\left(\chi(G(n,p)) \le \frac{d}{2\log d}\left(1+\frac{29.5\log\log d}{\log d}\right)\right) > 0.5$$

so, for such $d(n)$, a.s.

$$\chi(G(n,p)) \le \frac{d}{2\log d}\left(1+\frac{29.5\log\log d}{\log d}\right) + 5 \le \frac{d}{2\log d}\left(1+\frac{30\log\log d}{\log d}\right)\,.$$

This completes the proof of Theorem. ∎

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